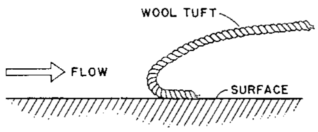

FIG. 1.

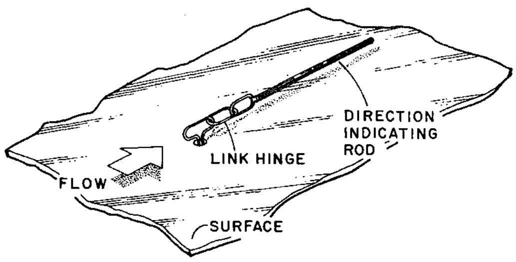

FIG. 2.

mental research for discovering regions of backflow, boundary-layer separation, and strong cross-flow, with the aim of correcting these conditions.

Commonly the tuft is a strand of wool, fixed at one end to the solid surface. Mechanically it is a cantilever beam with one end free and the other end fixed, and it assumes a position as in Fig. 1, exhibiting the flow direction at some distance from the surface. For the tuft to lie very close to the surface, as one would normally prefer, the bending stiffness must be exceedingly small. But this usually requires a fiber so fine that it is hard to see. Thus the requirements of flexibility and of ease of viewing are in conflict.

The ideal tuft would consist of two parts: a perfectly flexible universal hinge, and a rod-like direction indicator large enough to be easily visible and of a material neutrally buoyant in the fluid. Fig. 2 shows an approximation to this. The hinge is basically two or three links of a chain, made of very fine wire or nylon thread. The rod, for air, could be a wool tuft of suitable thickness; for water, it could be a dowel of wood or plastic having a specific gravity close to that of water.

In practice, flow-indicating tufts of this design have been very successful. The tufts move about freely and lie very close to the surface.

Thermal Stresses in an Elastic Half-Space With a Moving Boundary

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THE THERMAL STRESSES in an elastic semi-infinite solid whose boundary moves as a result of melting or freezing are considered in this note. The analysis is based on the uncoupled theory of thermal stresses. Such effects as viscosity and the variation of material properties with the temperature are also neglected. There is, of course, a question as to whether the elastic equations apply in phase transition conditions; the objective here is the investigation of the effect of inertia and comparing the result with quasi-static theory. Generally speaking, the neglect of inertia for problems *not involving a moving boundary* has been shown to be justified in Ref. 1, where a complete account of the various investigations is given. It will be shown that inertia plays an important role in moving boundary problems, and that its effect should be considered in certain cases.

Let us consider a solid occupying the region $x \geq 0$, initially unstressed and at a uniform temperature which shall be referred to as zero. The solid is to be melted by the application of heat at the plane $x = 0$. We assume that the solution of the temperature $T(x,t)$ and the free boundary position $s(t)$ is known. The thermal stresses σ_x, σ_y, and σ_z in directions x, y, and z are given in terms of the x-component of strain $e(x, t)$ by the equations

$$\left.\begin{aligned} \sigma_x &= \rho(c^2 e - \kappa T) \\ \sigma_y = \sigma_z &= (1 - \nu)^{-1}(\nu\sigma_x - E\alpha T) \end{aligned}\right\} \quad (1)$$

where E, ν, ρ, and α are Young's modulus, Poisson's ratio, density, and coefficient of thermal expansion, respectively, $\kappa = E\alpha(1 - 2\nu)^{-1}\rho^{-1}$, and c is the velocity of dilational waves. The strain $e(x, t)$ is the solution to the following boundary-value problem

$$c^2 e_{xx} - e_{tt} = \kappa T_{xx} \quad \text{in } R\text{:} \quad s(t) < x < \infty,\ t > 0 \quad (2)$$

$$e(s(t), t) = \kappa c^{-2} T_m \quad (3)$$

$$e(\infty, t) = 0 \quad (4)$$

$$e(x, 0) = e_t(x, 0) = 0 \quad (5)$$

Here T_m is the melting temperature and condition (3) states that at the boundary $x = s(t)$ the stress should be zero.

To obtain the solution, we examine the state of affairs in the x-t plane. In those cases when $s(t) \sim t^{1/2}$ the region of interest R of the x-t plane is bounded by the curve $x = s(t)$ and the line $t = 0$. This region is divided into two subregions R_1 and R_2. The region R_1 is the locus of all points whose domain of dependence fall entirely within R_1 and do not intersect the moving boundary. Region R_2 is simply $R - R_1$. The two regions are separated by a semi-infinite characteristic which is tangent to the curve $x = s(t)$ at the point $(\bar{x}, \bar{t})$ and extends from this point in the direction of increasing x and t.

Thus, in region R_1 the solution e is equal to the function $e_0(x, t)$ which satisfies Eq. (2) and initial conditions (5) and is regular as $|x| \rightarrow \infty$. This solution may be obtained either by transform technique or from the formula

$$e_0 = \frac{\kappa}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} T_{xx}(\xi, \tau)\, d\xi\, d\tau \quad (6)$$

In region R_2 the initial conditions no longer matter, and the solution may be chosen as

$$e = e_0(x, t) + f(ct - x) \quad (7)$$

Application of the boundary condition (3) results in the functional relationship

$$f(\xi) = \kappa c^{-2} T_m - e_0(s(t(\xi)), t(\xi)) \qquad \text{where } \xi = ct - s(t) \quad (8)$$

This equation determines f for any argument ξ once t has been solved in terms of ξ from the second equation in (8). Thus, the solution is completely determined in terms of $e_0(x, t)$ and the stresses may be found from Eq. (1). We note that the stress is discontinuous across the characteristic that separates the two regions by the constant amount $\Delta\sigma_x = \rho c^2 f(-x_0)$, where x_0 is the intersection of this characteristic with the x axis.

We also note that for times corresponding to the portion $0 < t < \bar{t}$ of the boundary the stress-free condition cannot be met and the initial conditions predetermine the boundary stress. This is a consequence of the initially infinite speed of the boundary, and the conjecture is made that if the effect of thermoelastic coupling is included, the speed of the boundary will be less than the sound speed of the medium. This difficulty does not arise in freezing problems, in which case the boundary moves opposite to the direction of wave propagation.

We shall now apply the solution obtained above to the case of a semi-infinite solid which is being melted by keeping the plane $x = 0$ at constant temperature T_1, $(T_1 > T_m)$. This is the Neumann problem and its solution is[2]

$$T = T_m \operatorname{erfc}(x/2\sqrt{kt})/\operatorname{erfc} h,\ s = 2h(kt)^{1/2} \quad (9)$$

$$\frac{K'(T_1 - T_m)\exp(-h^2k/k')}{k'^{1/2}\operatorname{erf}(h\sqrt{k/k'})} - \frac{KT_m \exp(-h^2)}{k^{1/2}\operatorname{erfc} h} = \pi^{1/2}k^{1/2}L\rho h \quad (10)$$

Here K is the conductivity, k the diffusivity, L the latent heat of fusion, primed quantities refer to the thermal constants of the molten solid, and the constant h is the root of the transcendental equation (10). If we introduce dimensionless coordinates $\xi = cx/4h^2k$ and $\tau = c^2t/4h^2k$, then the boundary curve becomes $\xi = \tau^{1/2}$, and the characteristic that separates regions R_1 and R_2 becomes $\xi = \tau + 1/4$. The steps outlined in Eqs. (6) to (8) may now be carried out. The result is as follows:

$$\sigma_x = -\tfrac{1}{2}\rho\kappa T_m(\operatorname{erfc} h)^{-1}[\exp h^2(\tau - \xi)\cdot\operatorname{erfc} h(\xi\tau^{-1/2} - 2\tau^{1/2}) + \exp h^2(\tau + \xi).\operatorname{erfc} h(\xi\tau^{-1/2} + 2\tau^{1/2})] \quad \text{for } 0 < \tau < 1/4,\ \xi > \tau^{1/2} \text{ and for } \tau > 1/4,\ \xi > \tau + 1/4 \quad (11)$$

$$\sigma_x = -\tfrac{1}{2}\rho\kappa T_m(\operatorname{erfc} h)^{-1}\{\exp h^2(\tau - \xi)\cdot[\operatorname{erfc} h(2\tau^{1/2} - \xi\tau^{-1/2}) - \operatorname{erfc} h\sqrt{1 - 4(\xi - \tau)}] + \exp h^2(\tau - \xi + 1 + \sqrt{1 - 4(\xi - \tau)})\cdot\operatorname{erfc} h(2 + \sqrt{1 - 4(\xi - \tau)}) - \exp h^2(\tau - \xi)\operatorname{erfc} h(\xi\tau^{-1/2} + 2\tau^{1/2})\} \quad \text{for } \tau > 1/4,\ \tau^{1/2} < \xi < \tau + 1/4 \quad (12)$$

The jump in the value of stress which propagates with velocity c is given by

$$\Delta\sigma_x = -\tfrac{1}{2}\rho\kappa T_m(\operatorname{erfc} h)^{-1}[\exp(-h^2/4) + \exp(3h^2/4)\cdot\operatorname{erfc}(2h)] \quad (13)$$

We note that this discontinuity in the value of stress increases with increasing h. The same effect is present in the expressions for σ_x as given in Eqs. (11) and (12). For a given material the coefficient $\rho\kappa T_m$ that appears in Eqs. (11) to (13) is a constant, whereas from Eq. (10) h increases as T_1 increases. This clearly demonstrates that under these circumstances the effect of inertia becomes important for all time, and its neglect is not justified. It may easily be verified that the quasi-static solution of the problem is given by $e = \kappa c^{-2}T$, $\sigma_x \equiv 0$, and $\sigma_y = \sigma_z = -\rho\kappa T$.

References

[1] Boley, B. A., and Weiner, J. H., *Thermal Stresses*, p. 59; New York, John Wiley and Sons, Inc., 1950.

[2] Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*, p. 287; Oxford University Press, 1959.

Interface Stability in a Nonuniform Acceleration Field

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The stability of two fluids separated by an initially plane interface and subject to a constant, normal acceleration field was considered by Taylor.[1] Corrections were introduced later to Taylor's analysis to account for the viscous and cohesive forces.[2]

The interface stability of two fluids in motion parallel to the common interface in a normal constant gravitational field was first studied by Helmholtz.[3, 4]

In some cases of practical importance the fluids are exposed to the action of nonuniform body-force distributions. For example, if a binary fluid with a distinguishable separation surface is in a vortex-type motion, the radial variation of the azimuthal velocity in the potential-flow region causes a spatially nonuniform centripetal acceleration field on both sides of the interface. The inclusion of viscosity effects would increase this nonuniformity.

The question is: how does the acceleration-field distribution affect the stability of the separation surface of two fluids?

The aim of this note is to show that the interface stability does not depend upon the spatial distribution of the acceleration field; the Helmholtz and Taylor stability criteria are directly applicable if the direction and the magnitude of the acceleration at the interface are known.

For brevity, only the invariance of the Taylor stability criterion will be shown here. Similar considerations can be applied also to Helmholtz's analysis.

Generalization of Taylor's Analysis

For the notation and the model considered here the reader is referred to Refs. 1 and 2.

The linearized equations describing the disturbed flow field are

$$\left.\begin{aligned} \partial u/\partial x + \partial v/\partial y &= 0 \\ \partial u/\partial t + (1/\rho)(\partial p/\partial x) &= 0 \\ \frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial y} + g(y) &= 0 \end{aligned}\right\} \quad (1)$$

and the interface is given by $y = \eta(x, t)$. Here, u and v are velocity components generated by the initially small disturbances; $g(y)$ is an arbitrary function of the coordinate perpendicular to the interface and it describes the body-force distribution throughout the fluid masses.

The above equations can be satisfied by

$$\left.\begin{aligned} u = \partial\phi/\partial x \quad v &= \partial\phi/\partial y \\ p = p_0 - \rho\int_0^y g(y)dy &- \rho\frac{\partial\phi}{\partial t} \end{aligned}\right\} \quad (2)$$

where p_0 is the mean pressure at the interface. Let us assume now

$$\left.\begin{aligned} \phi_1 &= A_1 \exp[-ky + \sigma t + ik'x] \\ \phi_2 &= A_2 \exp[ky + \sigma t + ik'x] \end{aligned}\right\} \quad (3)$$

where the subscripts 1, 2, refer to the upper and lower fluids, respectively.

The condition of pressure continuity at the disturbed interface yields the following expression:

$$\int_0^{y=\eta}[\rho_2 g_2(y) - \rho_1 g_1(y)]dy = \sigma[\rho_1 A_1 e^{-k\eta} - \rho_2 A_2 e^{k\eta}]\exp[\sigma t + ik'x] \quad (4)$$

If the initial disturbance amplitude is small compared with the wavelength, the time derivative of Eq. (4) can be written as

$$[\rho_2 g_2(\eta) - \rho_1 g_1(\eta)](\partial\eta/\partial t) = \sigma^2(A_1\rho_1 - A_2\rho_2)\exp[\sigma t + ik'x] \quad (5)$$

But

$$g_2(\eta) = g_1(\eta) = g_0 \quad (6)$$

where g_0 is the magnitude of the body forces at the interface.

The condition of continuous velocity at the interface for small initial-amplitude values yields

$$A_2 = -A_1 = -A \quad (7)$$

If one applies now the kinematic surface condition[5]

$$(\partial\eta/\partial t) + u(\partial\eta/\partial x) = v \quad (8)$$

and neglects the higher-order convective term,

$$\frac{\partial\eta}{\partial t} \simeq \frac{\partial\phi}{\partial y} = -kA\exp(\sigma t + ik'x) \quad (9)$$

The combination of Eqs. (5), (6), (7), and (9) yields an expression for the damping factor σ:

$$\sigma^2 = -kg_0[(\rho_2 - \rho_1)/(\rho_2 + \rho_1)] \quad (10)$$

The result is identical with that obtained for a constant acceleration field g_0.[1]

Thus, the stability of the interface depends upon the body forces

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